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IRREDUCIBLE ACTIONS AND FAITHFUL ACTIONS OF HOPF ALGEBRAS

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ABSTRACT

Let H be a Hopf algebra acting on an algebra A. We will examine the relationship between A, the ring of invariants A^H, and the smash product $A \# H$. We begin by studying the situation where A is an irreducible $A \# H$ module and, as an application of our first main theorem, show that if D is a division ring then $[D : D^H] \leqq \dim H$. We next show that prime rings with central rings of invariants satisfy a polynomial identity under the action of certain Hopf algebras. Finally, we show that the primeness of $A \# H$ is strongly related to the faithfulness of the left and right actions of $A \# H$ on A.

1. Introduction and definitions

Let H be a Hopf algebra acting on an algebra A. In this paper we examine the relationship between A, the ring of invariants A^H, and the smash product $A \# H$. In Section 2 we are primarily concerned with the case where A is an irreducible $A \# H$ module. The commuting ring of such an action is A^H and, by Schur's lemma, A^H is a division ring. We show that if A has finite left Goldie rank and H is finite dimensional then $[A : A^H] \leqq \dim H$. If A is a division ring then any action of $A \# H$ on A is irreducible, thus for any division ring D,

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$[D:D^H] \leqq \dim H$. This generalizes the results for finite groups acting on division rings [J 56] and for Hopf algebras acting on fields [S 68]. We also obtain some analogous results for certain infinite dimensional H which act in a finite dimensional fashion on A. By this we mean that the canonical image of H in End(A) is finite dimensional. In particular, this includes the action of algebraic automorphisms as well as finite dimensional Lie algebras of algebraic derivations.

We next apply our theorem on irreducible actions to prove that prime rings with central rings of invariants must satisfy a polynomial identity under certain group and Lie algebra actions. We conclude Section 2 by giving examples of finite groups G acting on division rings D such that $[D:D^G] < |K|$, for any non-identity subgroup K of G.

In Section 3, H is finite dimensional and we examine the relationship between the primeness of $A\#H$ and the faithfulness of the action of $A\#H$ on A. In [CFM], a new right action of $A\#H$ on A is defined and it is shown that $A\#H$ is prime if and only if A^H is prime and both the left and right actions of $A\#H$ are faithful. Our first result is to show the equivalence of the following three conditions: (i) $A\#H$ is prime, (ii) the left action of $A\#H$ is faithful on all non-zero H-stable left ideals of A, and (iii) the right action of $A\#H$ is faithful on all non-zero H-stable right ideals of A. We then give an example which shows that even if A is prime, the primeness of $A\#H$ is not equivalent to the faithfulness of the left and right actions of $A\#H$ on A. However, we do show that the following are equivalent: (i) $A\#H$ is prime, (ii) the left action of $A\#H$ on A is faithful, and (iii) the right action of $A\#H$ on A is faithful; provided either A is a domain with $H = u(L)$ or every H-stable non-zero one-sided ideal of A intersects A^H non-trivially with A^H prime. We conclude this paper with an example of a division ring D acted on by a finite group G such that the group algebra kG acts faithfully on D, but the smash product $D\#kG$ does not act faithfully on D.

At this point we introduce the notation that we will use throughout this paper. For a more thorough introduction to the terminology of Hopf algebras we recommend the books [A 80], [S 69a], and the paper [CF 86]. Throughout this paper k will be a field, H a Hopf algebra over k, and A an algebra over k. We let $\Delta: H \to H \otimes H$ be the comultiplication of H, $\varepsilon: H \to k$ is the counit of H, and $S: H \to H$ is the antipode of H. We will make frequent use of the sigma notation for Δ as in [S 69a]. All tensor products will be taken over k.

We say that A is a (left) H-module algebra if A is a left H-module such that (i) $h \cdot ab = \Sigma_{(h)}(h_{(1)} \cdot a)(h_{(2)} \cdot b)$ and (ii) $h \cdot 1_A = \varepsilon(h)1_A$, for all $a, b \in A$ and

$h \in H$. When A is an H-module algebra we can form the *smash product* $A \# H$. As a k vector space $A \# H$ is $A \otimes H$, however it has multiplication $(a \# h)(b \# l) = \Sigma_{(h)} a(h_{(1)} \cdot b) \# h_{(2)} l$, for all $a, b \in A$ and $h, l \in H$. $A \# 1$ and $1 \# H$ are subalgebras of $A \# H$ and when $a, b \in A$ and $h \in H$ we will often let the expression ahb be a shorthand for $(a \# 1)(1 \# h)(b \# 1)$. The *ring of invariants* A^H is defined as $\{a \in A \mid h \cdot a = \varepsilon(h)a, \text{ for all } h \in H\}$. If B is a subset of A such that $h \cdot B \subseteq B$, for all $h \in H$, then we say that B is H-stable.

When H is finite dimensional it contains one dimensional subspaces $\int_l$ and $\int_r$ known, respectively, as left and right integrals [S 69b]. They have the property that

$$\int_l = \{t \in H \mid ht = \varepsilon(h)t, \text{ for all } h \in H\}$$

and

$$\int_r = \{t \in H \mid th = \varepsilon(h)t, \text{ for all } h \in H\}.$$

Throughout we let $0 \neq t \in \int_l$ then t plays an important role as it induces an A^H bimodule *trace map* $\hat{t}: A \to A^H$ defined as $\hat{t}(a) = t \cdot a$, for all $a \in A$. If $h \in H$ then $th \in \int_l$ and since $\int_l$ is one dimensional, there exists $\lambda: H \to k$ such that $th = \lambda(h)t$, for all $h \in H$.

If A is an H-module algebra then A is not only a left $A \# H$ module, but using the action defined in [CFM], it is also a right $A \# H$ module. The left and right actions of $A \# H$ on A have the following definitions and notations:

$$\textit{left action}: (a \# h) \mapsto b = a(h \cdot b)$$

and

$$\textit{right action}: b \leftarrow\!\shortmid (a \# h) = S^{-1}(h^\lambda) \cdot ba,$$

for all $a, b \in A$ and $h \in H$, where $h^\lambda = \Sigma_{(h)} h_{(1)}\lambda(h_{(2)})$.

It will occasionally be useful to "reverse" the order of elements in $A \# H$. This can be done by using the following formula: $ah = \Sigma_{(h)} h_{(2)}(S^{-1}(h_{(1)}) \cdot a)$, for all $a \in A$ and $h \in H$.

If A is an H-module algebra then there is a natural k algebra homomorphism $\pi: H \to \mathrm{End}_k(A)$ given by $(\pi(h))(a) = h \cdot a$, for all $a \in A$ and $h \in H$. If $\dim_k H = m$ then clearly $\dim_k(\mathrm{Im}\, \pi) \leqq m$. However, even if H is infinite dimensional, it is possible that $\dim_k(\mathrm{Im}\, \pi) = n < \infty$. In this case we say that H acts *finitely of dimension* n and n is the *dimension of the action*.

We now briefly look at three particular types of Hopf algebras which will reoccur throughout this paper. If G is a group of k linear automorphisms of A then the group algebra $H = kG$ is a Hopf algebra and A is an H-module algebra. In this case we may denote the ring of invariants as A^G. Clearly, if $|G| = n$, then H is n dimensional. However even if G is infinite cyclic and is generated by an algebraic automorphism satisfying a polynomial of degree n, then H is infinite dimensional, but the dimension of the action is at most n.

If L is a Lie algebra over k acting on A as k linear derivations, then the universal enveloping algebra $H = U(L)$ is a Hopf algebra and A is an H-module algebra. Although $U(L)$ will be infinite dimensional, if L is finite dimensional and consists of algebraic derivations of A then the dimension of the action of $U(L)$ will be finite. More precisely, suppose $\{x_1, x_2, \ldots, x_m\}$ is a k basis of L and suppose each x_i is a derivation satisfying a polynomial of degree n_i. Then consider $\pi : U(L) \to \mathrm{End}_k(A)$; $\mathrm{Im}\,\pi$ is spanned by the images of the basis monomials given by the Poincaré–Birkhoff–Witt theorem. However, $\pi(x_i^{n_i})$ can be replaced by a linear combination of $\pi(x_i^j)$, with $j < n_i$. Thus $\dim_k(\mathrm{Im}\,\pi) \leqq \Sigma_{i=1}^m n_i$.

If char $k = p > 0$ and if L is a restricted Lie algebra acting on A as k linear derivations, then the restricted enveloping algebra $H = u(L)$ is a Hopf algebra and A is an H-module algebra. As opposed to the situation for $U(L)$, if L is n dimensional then $u(L)$ is p^n dimensional [J 62]. When a Lie algebra L acts on A, regardless of whether L is restricted, we may denote the ring of invariants as A^L. Many of the results in this paper will be for the special cases $H = kG$, $U(L)$, or $u(L)$.

2. Irreducible actions

In this section we consider the situation where A is an irreducible left $A \# H$ module. Analogous results can also be obtained by similar proofs if A is an irreducible right $A \# H$ module using the right action defined in Section 1. We begin with

LEMMA 2.1. (a) $A^H \cong \mathrm{End}_{A\#H}(A)$.
(b) *If A has no non-trivial H-stable left ideals then A^H is a division ring.*

PROOF. (a) This is actually Lemma 2.5 of [BM 86].
(b) Since A has no non-trivial H-stable left ideals, A is an irreducible left $A \# H$ module. Therefore, by Schur's lemma and (a), A^H is a division ring.

In Section 1 we said that a Hopf algebra H acts finitely of dimension n on A if the image of H in $\mathrm{End}_k(A)$ is n dimensional. Such actions therefore include all finite dimensional Hopf algebra actions as well as the action of certain infinite dimensional Hopf algebras. It is in this context in which we prove the main result of this section.

THEOREM 2.2. *Let A be a left H-module algebra such that $A \# H$ acts irreducibly on A, A has finite left Goldie rank, and H acts finitely of dimension n on A. Then $[A : A^H]_r \leqq n$, where $[A : A^H]_r$ is the dimension of A as a right vector space over the division ring A^H.*

PROOF. Let $M = \{w \in A \# H \mid w \to A = 0\}$; since $A \# H$ acts irreducibly on A, it follows that $A \# H/M$ acts faithfully and irreducibly on A. Therefore, we can apply the Jacobson Density Theorem to the action of $A \# H/M$ on A and, by Lemma 2.1(a), the commuting ring of this action is the division ring A^H. Furthermore, if $f \in A \# H$ we can identify the action of f on A with the action of $f + M \in A \# H/M$ on A.

Since the image of the map $\pi : H \to \mathrm{End}_k(A)$ has dimension n, there exist $h_1, h_2, \ldots, h_n \in H$ such that $\{\pi(h_1), \pi(h_2), \ldots, \pi(h_n)\}$ is a basis for $\pi(H)$. Let $T = (A \# h_1) + (A \# h_2) + \cdots + (A \# h_n)$, since the h_i are linearly independent, T is a free left A submodule of $A \# H$ of rank n. Now suppose $h \in H$, then there exist $\alpha_i \in k$ such that $\pi(h) = \Sigma_{i=1}^n \alpha_i \pi(h_i)$ and therefore if $a, b \in A$ we have

$$(a \# h) \to b = \left(\sum_{i=1}^{n} \alpha_i a \# h_i \right) \to b.$$

Thus, we note that if $f \in A \# H$ then there exists $f' \in T$ such that $f' \to b = f \to b$, for all $b \in A$. Furthermore the right A^H module map $\theta : T \to A \# H/M$ given by $\theta(t) = t + M$, for all $t \in T$, is now onto and has kernel $T \cap M$.

Suppose $S = \{e_i\}_{i=1}^m$ is a set of m elements of A which are right linearly independent over A^H, where m may be infinite. By Jacobson density and by our earlier observation about T, for any positive integer j not exceeding m, there exist $f_j \in T$ such that $f_j \to e_j = 1$ and $f_j \to e_i = 0$, all $i < j$. Now suppose there exist $a_i \in A$ such that $\Sigma_{i \geqq 1} a_i f_i = 0$, with some $a_i \neq 0$. If $t \geqq 1$ is the smallest subscript such that $a_t \neq 0$, then since $f_i \to e_t = 0$ for all $i > t$, we have

$$0 = \left(\sum_{i \geqq 1} a_i f_i \right) \to e_t = \left(\sum_{i \geqq t} a_i f_i \right) \to e_t = a_t + \left(\sum_{i > t} a_i f_i \right) \to e_t = a_t,$$

a contradiction. Therefore if $W = Af_1 + Af_2 + \cdots$, then W is a free left A submodule of T. As a left A module, $T \cong \bigoplus_{i=1}^n A$, hence T also has finite left Goldie rank. Therefore every free submodule of T must have finite rank, hence W must have finite rank. Thus the set S must be finite and so $[A : A^H]_r$ also must be finite.

If we now let $m = [A : A^H]_r$ then $A \# H/M \cong (A^H)_m$, the $m \times m$ matrices over A^H. Since $A \# H/M \cong T/(T \cap M)$, as right A^H modules, we now have

$$m^2 = [A \# H/M : A^H]_r = [T/(T \cap M) : A^H]_r \leqq [T : A^H]_r = mn,$$

hence $[A : A^H]_r = m \leqq n$.

The use of Jacobson Density in the proof of Theorem 2.2 is similar in spirit to the argument used in [S 68] for Hopf algebras acting on fields. The case where $[A : A^H]_r = \dim_k H$ is studied extensively in [CFM]. In particular, it is shown that $[A : A^H]_r = \dim_k H$ is equivalent to A/A^H being H^*-Galois. If $A = D$ is a division ring, then A has Goldie rank one and $A \# H$ always acts irreducibly on A. We record this important special case as

Corollary 2.3. *Let D be a left H-module algebra where D is a division ring and H is finite dimensional. Then $[D : D^H]_r \leqq \dim_k H$.*

The result of Corollary 2.3 is well-known when $H = kG$, G a finite group. In fact, when $H = kG$ it is also known [J 56] that $[D : D^H]_r = [D : D^H]_l$. For arbitrary finite dimensional Hopf algebras equality also holds [CFM] if either $[D : D^H]_r$ or $[D : D^H]_l$ equals $\dim_k H$. However, in general it is not known if equality must always hold, therefore we ask

Question 2.4. *If D is an H-module algebra where D is a division ring and H is a finite dimensional Hopf algebra, must $[D : D^H]_r$ equal $[D : D^H]_l$?*

As pointed out in Section 1, certain infinite groups and finite dimensional Lie algebras correspond to actions of infinite dimensional Hopf algebras which act in a finite dimensional fashion. In light of this, as a special case of Theorem 2.2 we now have

Corollary 2.5. (a) *Let D be a division ring and let g be an automorphism of D which is k linear and algebraic of degree n over k. Then $[D : D^G]_r \leqq n$, where G is the infinite cyclic group generated by g.*

(b) *Let D be a division ring acted on by a finite dimensional Lie algebra L of k linear derivations which are algebraic over k. Then $[D : D^L]_r \leqq n$ where n is the dimension of the action of $U(L)$ on D.*

Suppose H acts finitely of dimension n on A; if $H = kG$ it is easy to see that every essential left ideal of A contains an H-stable essential left ideal. Furthermore, by an argument of Popov [P 83], if $H = u(L)$ or $U(L)$ and A is (left) non-singular then essential left ideals of A again contain H-stable essential left ideals.

Now if A is also an irreducible left $A \# H$ module then A has no non-trivial H-stable left ideals, hence A contains no non-trivial essential left ideals. However, this is equivalent to A being semisimple Artinian [L 66]. Thus A would have finite Goldie rank and in these cases we can drop the finite Goldie rank assumption in Theorem 2.2. We record this as

COROLLARY 2.6. *Let A be a left H-module algebra such that $A \# H$ acts irreducibly on A and H acts finitely of dimension n on A, where either $H = u(L)$ or $U(L)$ with A (left) non-singular or $H = kG$. Then $[A : A^H]_r \leqq n$.*

Since we can drop the Goldie rank assumption from Theorem 2.2 in certain cases we now ask

QUESTION 2.7. *Does the conclusion of Theorem 2.2 still hold without the assumption that A has finite left Goldie rank?*

We will now apply Corollary 2.6 to obtain a result on group actions and Lie algebra actions with central rings of invariants.

THEOREM 2.8. *Suppose A is a prime algebra over a field k such that A is an H-module algebra where either*

(a) *$H = kG$ with G a finite p-nilpotent group and* char $k = p > 0$,

(b) *$H = u(L)$ with L a finite dimensional restricted nilpotent Lie algebra and* char $k = p > 0$, *or*

(c) *$H = U(L)$ with L a finite dimensional nilpotent Lie algebra of algebraic derivations of A.*

If A^H is central, then A satisfies a polynomial identity. Furthermore, if H is n dimensional then A satisfies an identity of degree $2[\sqrt{n}]$, where $[\sqrt{n}]$ is the greatest integer in $\sqrt{n}$.

PROOF. Since the non-zero elements of A^H are central and regular in A, we can localize to obtain a new ring S which is also acted on by H. If $H = kG$ and if $a\alpha^{-1} \in S^H$, with $a \in A$ and $\alpha \in A^H$, then $a\alpha^{-1} = g(a\alpha^{-1}) = g(a)\alpha^{-1}$, for all $g \in G$. Hence $a \in A^H$. Similarly, if $H = u(L)$ or $H = U(L)$ and if $a\alpha^{-1} \in S^H$ with $a \in A$ and $\alpha \in A^H$, then $0 = x(a\alpha^{-1}) = x(a)\alpha^{-1}$, for all $x \in L$. Thus $a \in A^H$ and we see that in all three cases S^H is the quotient field of A^H. As a result, when H acts on S all the hypotheses of the theorem still hold, but in addition S^H is a subfield of $Z(S)$.

In all three cases, every H-stable non-zero one-sided ideal of S intersects S^H

non-trivially. For $H = kG$ this is a result of [BI 73] and for $H = u(L)$ and $H = U(L)$ this is a result of [B 88]. Since S^H is a field containing the unit of S, it follows that if $\lambda \neq 0$ is any H-stable left ideal of S, then $1 \in S^H = \lambda \cap S^H \subseteq \lambda$ and so $\lambda = S$. Therefore the left action of $S \# H$ on S is irreducible.

Every ideal $I \neq 0$ of S contains a non-zero H-stable ideal [B 88], hence $I \cap S^H \neq 0$ and so S is a simple ring with 1. In particular, S must be (left) non-singular and by Corollary 2.6, $[S : S^H] \leqq m$, where m is the dimension of the action of H on both A and S. Therefore S and A satisfy a polynomial identity. If H is n dimensional, since $S^H \subseteq Z(S)$ we have $[S : Z(S)] \leqq [S : S^H] \leqq n$. However, every central simple algebra of degree l^2 over its center satisfies a polynomial identity of degree $2l$. Therefore it follows that S and A satisfy polynomial identities of degree $2[\sqrt{n}]$, where $[\sqrt{n}]$ is the greatest integer in $\sqrt{n}$.

We should note that the argument used in the proof of Theorem 2.8 also applies to the action of finite groups G where $|G|^{-1} \in k$ and also to the action of algebraic automorphisms. However, we did not mention such actions as part of the Theorem because stronger results are possible in both cases without using Theorem 2.8 ([K 74], [Br], [O 81]). On the other hand, the results in Theorem 2.8 are in some sense best possible as the hypotheses that G be p-nilpotent and L be nilpotent cannot be weakened to solvability, nor can the hypothesis that A^H be central be weakened to A^H commutative. This can be seen in the following set of examples, which are based on an example of Bergman and Kharchenko for group actions.

Example 2.9. Let $R = k[x, y]$ be the non-commutative free algebra in two variables over the field k and let $A = M_2(R)$ be the 2×2 matrices over R.

Let h_1 and h_2 be the inner automorphisms of A induced by conjugation by, respectively,

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$$

and let g be the inner automorphism of A induced by conjugation by $\left[\begin{smallmatrix} \omega & 0 \\ 0 & 1 \end{smallmatrix}\right]$, where $\omega \in k$, $\omega \neq 1$ is an nth root of 1. Now let K be the group generated by $\{h_1, h_2\}$ and let G be the group generated by $\{h_1, h_2, g\}$.

If char $k = p > 0$ then $|K| = p^2$, $|G| = np^2$, K is abelian, and G is solvable but not p-nilpotent. Furthermore,

$$A^K = \left\{ \begin{bmatrix} \alpha & s \\ 0 & \alpha \end{bmatrix} \middle| \alpha \in k, s \in R \right\} \quad \text{and} \quad A^G = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \middle| \alpha \in k \right\}.$$

Thus A is prime, K is finite and abelian, A^K is commutative, yet A does not satisfy a polynomial identity. In addition, A is prime, G is finite and solvable, A^G is central, yet A does not satisfy a polynomial identity. This shows, for group actions, the necessity of both the group being p-nilpotent and the invariants being central.

Analogously, let N be the Lie algebra of inner derivations of A induced by the elements of the set $\{[\begin{smallmatrix} 0 & \beta x + \gamma y \\ 0 & 0 \end{smallmatrix}] \mid \beta, \gamma \in k\}$ and let L be the Lie algebra of inner derivations of A induced by the elements of the set $\{[\begin{smallmatrix} \alpha & \beta x + \gamma y \\ 0 & 0 \end{smallmatrix}] \mid \alpha, \beta, \gamma \in k\}$. Then in this case N is 2 dimensional abelian, L is 3 dimensional solvable but not nilpotent,

$$A^N = \left\{ \begin{bmatrix} \alpha & s \\ 0 & \alpha \end{bmatrix} \middle| \alpha \in k, s \in R \right\}, \qquad A^L = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \middle| \alpha \in k \right\},$$

and both N and L act on A as algebraic derivations. Furthermore, A is prime, N is abelian, A^N is commutative, yet A does not satisfy a polynomial identity and, similarly, A is prime, L is solvable, A^L is central, yet A does not satisfy a polynomial identity. In addition if $\operatorname{char} k = p > 0$, then N and L are both restricted, thus we see the necessity for both the Lie algebra being nilpotent and the invariants being central in Theorem 2.8.

We now turn our attention to groups acting on division rings. If D is a division ring and G is a finite group acting on D, then $[D : D^G] \leqq |G|$. However, an example of Snider shows that even if G consists of $|G|$ distinct automorphisms, we can have $[D : D^G] < |G|$. In Snider's example, let D have characteristic 2 and let $x \in D$, $x \notin Z(D)$ such that $x^2 \in Z(D)$. Now let G be the group of automorphisms of D generated by the inner automorphisms induced by conjugation by x and $1 + x$. In this case $G \cong Z_2 \times Z_2$ and $[D : D^G] = 2 < 4 = |G|$. However, if K is any subgroup of G of order 2, then $[D : D^G] = [D : D^K] = |K|$. This raises the following question: if a finite group G acts on a division ring D, must G contain a subgroup K such that $[D : D^G] = [D : D^K] = |K|$? We answer this question in the negative with

Example 2.10. Let D be the real quaternions and let $\omega = e^{2\pi i/p}$, where $p > 2$ is any prime. If g is the inner automorphism of D induced by conjugation by ω and if G is the cyclic group generated by g then, since $\mathbf{C} = D^G$, we have

$$[D:D^G]=[D:\mathbf{C}]=2<p=|G|.$$

Since G is cyclic of prime order, it contains no subgroups of order 2.

3. Faithful actions and primeness

In this section we turn our attention to questions regarding the faithfulness of the action of $A\#H$ on A for finite dimensional H. In [CF 86] it is shown that if H has an S fixed integral then $A\#H$ being prime is equivalent to A being a faithful left and right $A\#H$ module and A^H being prime. In [CFM 90], this result is generalized by defining and using a slightly different right action. It is also shown in [C 87] that for $H=kG$, right faithfulness is equivalent to left faithfulness and it is asked in [CFM 90] if right faithfulness is always equivalent to left faithfulness, under either right action.

In our results, the right action of $A\#H$ on A will always be the one introduced in [CFM 90] and discussed in Section 1. In the following theorem, the proofs that (i)⇒(ii) and (i)⇒(iii) will follow from the fact [CFM 90] that $[A^H, A, A, A\#H]$ form a Morita context with maps $(\ ,\ ): A\otimes_{A\#H}A\to A^H$ via $(a,b)=t\cdot(ab)$ and $[\ ,\ ]: A\otimes_{A^H}A\to A\#H$ via $[a,b]=atb$, where t is a left integral for H.

The following theorem generalizes a result of McConnell and Sweedler on simple smash products [McS 71].

THEOREM 3.1. *Let A be a left H-module algebra, with H finite dimensional. Then the following are equivalent*:

(i) *$A\#H$ is prime.*

(ii) *Every H-stable non-zero left ideal of A is a faithful left $A\#H$ module.*

(iii) *Every H-stable non-zero right ideal of A is a faithful right $A\#H$ module.*

PROOF. (ii)⇒(i). Let I, J be non-zero ideals of $A\#H$; since $J\rightharpoonup A\neq 0$, let $a\in A$ be such that $J\rightharpoonup a\neq 0$. Since J is, in particular, a left ideal of $A\#H$ it follows that $J\rightharpoonup a$ is an H-stable left ideal of A. Therefore $0\neq I\rightharpoonup(J\rightharpoonup a)=IJ\rightharpoonup a$ and so $IJ\neq 0$. Thus $A\#H$ is prime.

(iii)⇒(i). An analogous argument to the one above holds. Briefly, let $b\in A$ such that $b\leftharpoonup I\neq 0$ and we have $0\neq(b\leftharpoonup I)\leftharpoonup J=b\leftharpoonup IJ$, thus $IJ\neq 0$ and $A\#H$ is prime.

(i)⇒(ii) and (i)⇒(iii). Let $\eta\neq 0$ be an H-stable left (or right) ideal of A. Then $[\eta, A]$ (or $[A,\eta]$) is a non-zero ideal of $A\#H$. If $x\in A\#H$ such that $x\rightharpoonup\eta=0$ (or $\eta\leftharpoonup x=0$) then $x[\eta,A]=[x\rightharpoonup\eta,A]=0$ (or $[A,\eta]x=[A,\eta\leftharpoonup x]=0$). Since $A\#H$ is prime, it follows that $x=0$.

Although Theorem 3.1 shows that primeness of $A \# H$ is equivalent to the faithfulness of the left and right actions of $A \# H$ on all non-zero H-stable left and right ideals of A, it is not generally the case, even when A is prime, that the primeness of $A \# H$ is equivalent to the faithfulness of the left and right actions of $A \# H$ on A. This can be seen in

EXAMPLE 3.2. Let A be any prime ring of characteristic 2 which is not a domain and let $a \in A$ such that $a^2 = 0$ and $a \neq 0$. Now let d be the inner derivation of A induced by a and let L be the one dimensional restricted Lie algebra containing d. Thus $H = u(L)$ is 2 dimensional and Aa and aA are non-zero H-stable left and right ideals of A which are annihilated by the left and right action of $ad \in A \# H$. Hence $A \# H$ is not prime. On the other hand, if the left action of $\alpha + \beta d \in A \# H$ annihilates A, then for $r, s \in A$ we have

$$0 = (\alpha + \beta d)\cdot(rs) = \alpha rs + \beta d(rs) = \alpha rs + \beta d(r)s + \beta rd(s)$$

and

$$0 = ((\alpha + \beta d)\cdot(r))s = (\alpha r + \beta d(r))s = \alpha rs + \beta d(r)s.$$

Subtracting the second equation from the first yields $\beta rd(s) = 0$, hence $\beta d(A)A = 0$. Therefore $\beta = 0$ and it immediately follows that $\alpha = 0$. Thus the left action of $A \# H$ is faithful on A and an analogous argument works for the right action of $A \# H$ on A.

Despite the previous example, there are several interesting cases where the primeness of $A \# H$ is equivalent to the faithfulness of the action of $A \# H$ on A. In order to prove our main result in that direction, we first need

LEMMA 3.3. *Let A be a left H-module algebra with H finite dimensional, and assume that A is a faithful left (right) $A \# H$ module. Then every non-zero H-stable left (right) ideal is a faithful left (right) $A \# H$ module, in either of the following situations:*

(1) $H = u(L)$ and A is a domain,

(2) A^H is prime and every H-stable non-zero one-sided ideal of A intersects A^H non-trivially.

PROOF. For case (1), we note that whenever an element of $A \# u(L)$ acts on A, regardless of whether it is a left or right action, it acts as a sum and composition of derivations, left multiplications, and right multiplications. However, in [B 89] it is shown that any such endomorphism of a domain which annihilates a non-zero one-sided ideal must annihilate the entire ring.

Since $A \# H$ acts faithfully on A, it must also act faithfully on all non-zero one-sided ideals of A.

For case (2), suppose $A \# H$ acts faithfully on A on the left and let $\eta \neq 0$ be an H-stable left ideal of A. Let $M = \eta \cap A^H$ and we consider $\lambda = \{a \in A \mid aM = 0\}$ and $\lambda \cap A^H$; since A^H is prime we have $0 = \lambda \cap A^H$. However, λ is an H-stable left ideal of A, thus $\lambda = 0$. Now suppose $w \in A \# H$; since $M \subseteq A^H$ we have $w \rightarrow AM = (w \rightarrow A)M$. Thus if $w \rightarrow \eta = 0$, it follows that $(w \rightarrow A)M = w \rightarrow AM \subseteq w \rightarrow \eta = 0$, hence $w \rightarrow A \subseteq \lambda = 0$. As a result, $w = 0$ and so $A \# H$ acts faithfully on η. An analogous argument works for the right action of $A \# H$ on H-stable right ideals of A.

We should point out that the condition in Lemma 3.3 that H-stable non-zero one-sided ideals of A intersect A^H non-trivially is satisfied quite often. In particular, if A is semiprime then this condition is satisfied in all of the following cases:

(i) $H = kG$; G a finite group with $|G|^{-1} \in k$ [BI 73],

(ii) $H = kG$; $G = \langle g \rangle$ and g is an algebraic automorphism of A [HN 75],

(iii) $H = kG$; G a finite p-nilpotent group and char $k = p > 0$ [BI 73],

(iv) $H = u(L)$; L a finite dimensional restricted nilpotent Lie algebra with char $k = p > 0$ [B 88],

(v) $H = U(L)$; L a finite dimensional nilpotent Lie algebra of derivations acting algebraically on A [B 88].

Even if A is not semiprime this condition is always satisfied whenever $A \# H$ is semiprime [CF 86].

By combining Theorem 3.1 and Lemma 3.3, we obtain the final main result of this paper.

COROLLARY 3.4. *Let A be a left H-module algebra with H finite dimensional. Then if either condition* (1) *or* (2) *of Lemma* 3.3 *is satisfied, the following are equivalent*:

(i) *$A \# H$ is prime.*

(ii) *$A \# H$ acts faithfully on A on the left.*

(iii) *$A \# H$ acts faithfully on A on the right.*

PROOF. By Theorem 3.1, we have (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii). However, by Lemma 3.3, if $A \# H$ acts faithfully on A on the left (right), then it also acts faithfully on all non-zero H-stable left (right) ideals of A. Thus, by Theorem 3.1, (ii)$\Rightarrow$(i) and (iii)$\Rightarrow$(i).

We conclude this paper by returning to the case where a finite group G acts on a division ring D as k linear automorphisms, where k is a subfield of $Z(D)$. An examination of the proof of Theorem 2.2 shows that $D\#kG$ acts faithfully on D if and only if $[D:D^G]=|G|$. In the Snider example mentioned in Section 2, not only is $[D:D^G]<|G|$, but the group algebra $Z(D)G$ fails to act faithfully on D as the trace map

$$t(r)=r+x^{-1}rx+(1+x)^{-1}r(1+x)+(x+x^2)^{-1}r(x+x^2)=0,$$

for all $r\in D$. This raises the question: if kG acts faithfully on D, must $D\#kG$ act faithfully on D? We answer this question in the negative with

EXAMPLE 3.5. Consider the special case in Example 2.10 where $p=3$. Thus $[D:D^G]=2<3=|G|$, hence $D\#kG$ does not act faithfully on D, where we let k be the real numbers. In particular, if $f=1+\omega g+\omega^2g^2\in D\#kG$, then

$$\begin{aligned} f\rightharpoonup r = r+\omega g(r)+\omega^2g^2(r) &= r+\omega(\omega^{-1}r\omega)+\omega^2(\omega^{-2}r\omega^2)\\ &= r(1+\omega+\omega^2)=0, \end{aligned}$$

for all $r\in D$. However, we claim that although $D\#kG$ does not act faithfully on D, kG does act faithfully on D.

Suppose $0\neq h\in kG$ such that $h(r)=0$, for all $r\in D$; then without loss of generality we may assume that $h=1+\alpha g+\beta g^2$, where $\alpha,\beta\in k$. If $r,s\in D$ then

$$(*)\qquad 0=h(rs)=rs+\alpha g(rs)+\beta g^2(rs)=rs+\alpha\omega^{-1}rs\omega+\beta\omega^{-2}rs\omega^2$$

and

$$(**)\qquad 0=h(r)s=rs+\alpha g(r)s+\beta g^2(r)s=rs+\alpha\omega^{-1}r\omega s+\beta\omega^{-2}r\omega^2s.$$

Therefore, by $(*)$, $-\omega^2rs=\alpha\omega rs\omega+\beta rs\omega^2$ and, by $(**)$, $-\omega^2rs=\alpha\omega r\omega s+\beta r\omega^2s$. Hence $\alpha\omega rs\omega+\beta rs\omega^2=\alpha\omega r\omega s+\beta r\omega^2s$ and so

$$(***)\qquad \alpha\omega r(s\omega-\omega s)=\beta r(\omega^2s-s\omega^2).$$

Since $\omega\notin k$ and $\omega^2\notin k$, there exists $s\in D$ such that $s\omega-\omega s\neq 0$ and $\omega^2s-s\omega^2\neq 0$. Therefore, by applying Martindale's theorem [M 69] to $(***)$, $\alpha\omega$ and β must be linearly dependent over k, hence either $\alpha=0$ or $\beta=0$. However, by $(***)$, α and β must both be 0, hence $h(r)=r=0$, for all $r\in D$, a contradiction. Thus kG acts faithfully on D.

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